PICARD ORBITS OF LIPSCHITZIAN TYPE MAPPINGS AND THEIR ACCUMULATION POINTS ON DISTANCE SPACES

Vasile BERINDE[1], prof. PhD
Mitrofan M. CHOBAN[2], academician, full prof.
[1]Technical University of Cluj-Napoca, North University Center at Baia Mare, Romania
[2]Tiraspol State University, Republic of Moldova

Abstract. We give a new example which illustrates the fact that some Picard orbits may have n distinct accumulation points, where n is a given natural number.
Keywords: distance space, N-distance space, F-distance space, H-distance space, quasi-metric space, contraction mapping, fixed point.

ORBITELE PICARD ALE APLICAŢIILOR DE TIP LIPSCHITZIAN ŞI PUNCTELE LOR DE ACUMULARE PE SPAŢII CU DISTANŢĂ

Rezumat. Construim un nou exemplu care ilustrează faptul că unele orbite Picard pot avea n puncte de acumulare distincte, unde n este un numar natural dat.
Cuvinte cheie: Spaţiu cu distanţă, spaţiu cu N-distanţă, spaţiu cu F-distanţă, spaţiu cu H-distanţă, spaţiu quasimetric, contracţie, punct fix.

1. Preliminaries

In [2] the authors proposed the following two problems.

Problem 1. *Let $g : X \longrightarrow X$ be a contraction of a complete quasimetric sace (X, d). Is it true that g have fixed points?*

Problem 2. *Let $g : X \longrightarrow X$ be a contraction of a complete F-symmetric sace (X, d). Is it true that g have fixed points?*

These two problems were solved in [8]. Our aim in the present paper is to present an example that illuminates the results in [2] and [8] to some extent. Distinct variants of the fixed point problem in general distance spaces were examined in [1, 2, 4, 5, 6, 7, 8, 15] and other articles.

Throughout the paper, by a space we understand a topological T_0-space, and we use the terminology from [9, 10, 14].

Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be a mapping such that for all $x, y \in X$ we have:

(i_m) $d(x, y) \geq 0$;

(ii_m) $d(x, y) + d(y, x) = 0$ if and only if $x = y$.

Then (X, d) is called a *distance space* and d is called a *distance* on X.

Let d be distance on X and let $B(x, d, r) = \{y \in X : d(x, y) < r\}$ be the *ball* with the center x and radius $r > 0$. The set $U \subset X$ is called *d-open* if for any $x \in U$ there exists $r > 0$ such that $B(x, d, r) \subset U$. The family $\mathcal{T}(d)$ of all d-open subsets is the topology on X generated by d. The space $(X, \mathcal{T}(d))$ is a T_0-space.

A distance space is a *sequential space*, i.e., a set $B \subseteq X$ is closed if and only if for any sequence $\{x_n\}$ in B, all limits of $\{x_n\}$ are in B [9].

Let (X, d) be a distance space, $\{x_n : n \in \mathbb{N} = \{1, 2, ...\}\}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n : n \in \mathbb{N}\}$:

1) is *convergent* to x if and only if $\lim_{n\to\infty} d(x, x_n) = 0$. We denote this by $x_n \to x$ or $x = \lim_{n\to\infty} x_n$.

2) is *Cauchy* or fundamental if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

We say that a distance space (X, d) is *complete* if every Cauchy sequence in X converges to some point in X.

Let d be a distance on X such that for all $x, y \in X$ we have:

(iii_m) $d(x, y) = d(y, x)$.

Then (X, d) is called a *symmetric space* and d is called a *symmetric* on X.

Let d be a distance on X such that for all $x, y, z \in X$ we have:

(iv_m) $d(x, z) \leq d(x, y) + d(y, z)$.

Then (X, d) is called a *quasimetric space* and d is called a *quasimetric* on X.

A distance d on a set X is called a *metric* if it is simultaneously a symmetric and a quasimetric.

2. Conditions of existence of fixed points

Let X be a non-empty set and $d(x, y)$ be a distance on X with the following property:

(N) for each point $x \in X$ and any $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that from $d(x, y) \leq \delta$ and $d(y, z) \leq \delta$ it follows $d(x, z) \leq \varepsilon$.

Then (X, d) is called an *N-distance space* and d is called an *N-distance* on X. If d is a symmetric, then we say that d is an D-symmetric (see [11, 12, 13, 16, 17]).

If d satisfy the condition

(F) for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that from $d(x, y) \leq \delta$ and $d(y, z) \leq \delta$ it follows $d(x, z) \leq \varepsilon$,

then d is called an *F-distance* or a *Fréchet distance* and (X, d) is called an *F-distance space* (see [3, 11]).

Remark. Any F-distance is an N-distance.

A distance space (X, d) is called an *H-distance space* if for any two distinct points $x, y \in X$ there exists $\delta = \delta(x, y) > 0$ such that $d(x, z) + d(y, z) \geq \delta$ for each point $z \in X$, i.e., $B(x, d, \delta) \cap B(y, d, \delta) = \varnothing$.

Remark. Any N-symmetric is an H-distance.

A space (X, d) is a H-distance space if and only if any convergent sequence has a unique limit point (see [11], Theorem 3).

Consider the mapping $\varphi : X \longrightarrow X$. and let $\varphi^1 = \varphi$ and $\varphi^{n+1} = \varphi \circ \varphi^n$ for each $n \in \mathbb{N}$ $= \{1, 2, ...\}$ be its iterates. If $x \in X$, then put $x_0 = x$ and consider $x_n = \varphi^n(x_0)$, for every $n \in \mathbb{N}$. The set $O(x, \varphi) = \{x_n : n \in \mathbb{N}\}$ is called the Picard orbit of the point x.

In the paper [8] the following assertions were established.

Theorem 2.1. Let d be an N-distance and an H-distance on a space X and let $\varphi : X \longrightarrow X$ be a mapping with the following properties:

(i) the mapping φ is continuous or there exists a number $\lambda > 0$ such that $d(\varphi(x), \varphi(y)) \leq \lambda \cdot d(x, y)$ for all points $x, y \in X$;

(ii) for some point $e \in X$ the Picard orbit $O(e, \varphi) = \{e_n = \varphi^n(e) : n \in \mathbb{N}\}$ has an accumulation point and $\lim_{n \to \infty} d(e_n, e_{n+1}) = 0$.

Then:

1. The mapping φ has fixed points. Any accumulation point of the orbit $O(e, \varphi)$ is a fixed point of φ.

2. The orbit of the point e has not periodic points.

3. If $\lim_{n \to \infty} d(g^n(y), g^{n+1}(y)) = 0$, for each point $y \in X$, then any periodic point of the mapping φ is a fixed point of φ.

4. The space $(X, \mathcal{T}(d))$ is first-countable and Hausdorff.

Corrolary 2.2. Let d be a quasimetric and an H-distance on a space X and let $\varphi : X \longrightarrow X$ be a mapping with properties:

(i) the mapping φ is continuous or there exists a number $\lambda > 0$ such that $d(\varphi(x), \varphi(y)) \leq \lambda \cdot d(x, y)$ for all points $x, y \in X$;

(ii) for some point $e \in X$ the Picard orbit $O(e, \varphi) = \{e_n = \varphi^n(e) : n \in \mathbb{N}\}$ has an accumulation point and $\lim_{n \to \infty} d(e_n, e_{n+1}) = 0$ and $\lim_{n \to \infty} d(e_n, e_{n+1}) = 0$.

Then:

1. The mapping φ has fixed points. Any accumulation point of the orbit $O(e, \varphi)$ is a fixed point of φ.

2. The orbit of the point e has no periodic points.

3. If $\lim_{n \to \infty} d(g^n(y), g^{n+1}(y)) = 0$ for each point $y \in X$, then any periodic point of the mapping φ is a fixed point of φ.

4. The space $(X, \mathcal{T}(d))$ is first-countable and Hausdorff.

Corrolary 2.3. Let d be a complete quasimetric and an H-distance on a space X and $\varphi : X \longrightarrow X$ be a mapping with properties:

(i) the mapping φ is continuous or there exists a number $\lambda > 0$ such that $d(\varphi(x), \varphi(y)) \leq \lambda \cdot d(x, y)$ for all points $x, y \in X$;

(ii) for each point $x \in X$ and the Picard orbit $O(x, \varphi) = \{x_n = \varphi^n(x) : n \in \mathbb{N}\}$ there exists a non-negative number $\mu(x) < 1$ such that $d(\varphi(x_n), \varphi(x_m)) \leq \mu(x) \cdot d(x_n, x_m)$ for all $n, m \in \mathbb{N}$.

Then:

1.The mapping φ has fixed points.

2.Any periodic point of the mapping φ is a fixed point of φ.

3.Any Picard orbit is a Cauchy convergent sequence to some fixed point of the mapping φ.

4.The space $(X, \mathcal{T}(d))$ is first-countable and Hausdorff.

Remark. The condition that d is an H-distance on X is essential (see [8]).

In connection with the above results it is important to answer the following question.

Problem 3. *Let (X, d) be a complete quasimetric space, where d is an H-quasimetric, $a \in X$ and let $g : X \longrightarrow X$ be a mapping such that $d(g^n(a), g^{n+1}(a)) > d(g^{n+1}(a), g^{n+2}(a))$ and $\lim_{n \to \infty} d(g^n(a), g^{n+1}(a)) = 0$.*

How many accumulations points does have the orbit of g at the point a?

In [8] it was constructed an example to answer Problem 3, where the orbit of g at the point a has two accumulation points.

The main aim of the next section is to show, by virtue of an appropriate example, that the orbit of a point may have actually m distinct accumulation points, where m is a given natural number.

3. An example of a Picard orbit possessing $m \geq 2$ accumulation points

Example 3.1. Let $m \geq 2$ and consider a set $B = \{b_1, b_2, ..., b_m\}$ with m distinct points. Assume that $B \cap \mathbb{N} = \varnothing$ and let $X = \mathbb{N} \cup B$. In $\mathbb{N}$ consider a sequence $\{i_{(n,1)}, i_{(n,2)}, ..., i_{(n,m)}, i_{(n,m+1)} : n \in \mathbb{N}\}$ such that:

(i) $4 = i_{(1,1)} < ... < i_{(1,m+1)} < i_{(2,1)} < ... < i_{(n-1,m+1)} < i_{(n,1)} < i_{(n,2)} < ... < i_{(n,m)} < i_{(n,m+1)} < i_{(n+1,1)} < ...$;

(ii) $\Sigma\{m^{-1} : m \in \mathbb{N}, i_{(n,i)} \leq m < i_{(n,i+1)}\} < 2$, $\Sigma\{m^{-1} : m \in \mathbb{N}, i_{(n,i)} \leq m \leq i_{(n,i+1)}\} \geq 2$ for each $n \in \mathbb{N}$ and $i \leq m$;

(iii) $\Sigma\{m^{-1} : m \in \mathbb{N}, i_{(n,m+1)} \leq m < i_{(n+1,1)}\} < 2$, $\Sigma\{m^{-1} : m \in \mathbb{N}, i_{(n,m+1)} \leq m \leq i_{(n+1,1)}\} \geq 2$, for each $n \in \mathbb{N}$.

Since $0 \notin \mathbb{N}$ and we need the numbers $i_{(n-1,m+1)}$ and $i_{(n,0)}$ for each $n \in \mathbb{N}$, it is convenient to put $i_{(0,m+1)} = 1$ and $i_{(n,0)} = i_{(n-1,m+1)}$.

Consider on $\mathbb{N}$ the function $f(n) = \Sigma\{m^{-1} : m \in \mathbb{N}, m \leq n\}$. The sets $I_{(n,i)} = \{k \in \mathbb{N} : i_{(n,i)} \leq k \leq i_{(n,i+1)}\}$, $I_{(n,m+1)} = \{k \in \mathbb{N} : i_{(n,m+1)} \leq k \leq i_{(n+1,1)}\}$ are called the m-intervals of integers of the rank n.

Now we construct on X the distance d with the conditions:

(C1) $d(x, x) = 0$ for each $x \in X$;

(C2) $d(b_i, b_j) = 1$ for all distinct $i, j \in \{1, 2, ..., m\}$;

(C3) $d(n, b_i) = 1$ for all $n \in \mathbb{N}$ and $i \in \{1, 2, ..., m\}$;

(C4) $d(n, m) = \min\{1, |f(n) - f(m)|\}$, for all $n, m \in \mathbb{N}$;

(C5) If $y, n \in \mathbb{N}$, $1 \leq i \leq m$, $x = b_i$ and $i_{(n-1,m+1)} \leq y \leq i_{(n,m+1)}$, then $d(x, y) = d(b_i, y)$ $= (i_{(n,i)})^{-1} + |f(y) - f(i_{(n,i)})|$.

By construction, $0 \leq d(x, y) \leq 1$, for all $x, y \in X$. Moreover, if $x, y \in \mathbb{N} \subset X$ and $d(x, y) < 1$, then we have three possibilities:

(i) There exists $n = n(x, y) \in \mathbb{N}$ such that $x, y \in [i_{(n-1,m+1)}, i_{(n,2)}]$;

(ii) There exists $n = n(x, y) \in \mathbb{N}$ such that $x, y \in [i_{(n,m)}, i_{(n+1,1)}]$;

(iii) There exists $n = n(x, y) \in \mathbb{N}$ and $i \leq m$ such that $i \geq 2$ and $x, y \in [i_{(n,i-1)}, i_{(n,i+1)}]$.

In the above cases, x, y are numbers belonging to an m-interval or to the union of two adjacent m-intervals.

We put $\varphi(b_i) = b_i$, for each $i \leq m$ and $\varphi(n) = n + 1$, for each $n \in \mathbb{N}$. By construction, $Fix\,(\varphi) = \{b_1, b_2, ..., b_m\}$. We prove the following claims.

Property 1. (X, d) *is a complete distance space.*

Proof. The space (X, d) has no non-trivial Cauchy sequences, i.e., if $\{x_n \in X : n \in \mathbb{N}\}$ is a Cauchy sequence, then there exists $k \in \mathbb{N}$ such that $x_k = x_n$ for all $n \geq k$ and $\lim_{n\to\infty} x_n = x_k$.

Property 2. (X, d) *is a quasimetric space.*

Proof. Fix three distinct points $x, y, z \in X$.

Case 1. $x, y, z \in \mathbb{N}$.

On $\mathbb{N}$ the distance d is a metric. Hence $d(x, z) \leq d(x, y) + d(y, z)$.

Case 2. $x, y, z \in B$.

On M the distance d is a discrete metric. Hence $1 = d(x, z) \leq d(x, y) + d(y, z) = 2$.

Case 3. $x, y \in B$ and $z \in \mathbb{N}$.

In this case $d(x, z) \leq 1 = d(x, y) < d(x, y) + d(y, z)$.

Case 4. $x, z \in B$ and $y \in \mathbb{N}$.

In this case $d(x, z) \leq 1 = d(y, z) < d(x, y) + d(y, z)$.

Case 5. $y, z \in B$ and $x \in \mathbb{N}$.

In this case $d(x, z) \leq 1 = d(y, z) < d(x, y) + d(y, z) = 2$.

Case 6. $y \in B$ and $x, z \in \mathbb{N}$.

In this case $d(x, z) \leq 1 = d(x, y) < d(x, y) + d(y, z)$.

Case 7. $z \in B$ and $x, y \in \mathbb{N}$.

In this case $d(x, z) = 1 = d(y, z) < d(x, y) + d(y, z)$.

For $x \in B$ and $y, z \in \mathbb{N}$ we consider the following cases.

Case 8. $x \in B$, $y, z \in \mathbb{N}$ and $d(y, z) = 1$.

In this case $d(x, z) \leq 1$, $d(x, y) \leq 1$ and $d(x, z) < d(x, y) + d(y, z)$.

Case 9. $x = b_i \in B$, $y, z \in \mathbb{N}$, $d(y, z) < 1$, $n \in \mathbb{N}$ and $y, z \in [i_{(n-1,m+1)}, \leq i_{(n,m+1)}]$.

In this case $d(x, z) = d(b_i, z) = min\{1, (i_{(n,1)})^{-1} + |f(z) - f(i_{(n,i)})|\} = .\ min\{1, (i_{(n,1)})^{-1} + |(f(z) - f(y)) + (f(y) - f(i_{(n,i)}))|\} \leq min\{1, (i_{(n,1)})^{-1} + |f(z) - f(y)| + |f(y) - f(i_{(n,i)})|\} \leq min\{1, (i_{(n,1)})^{-1} + |f(y) - f(i_{(n,i)})|\} + |f(z) - f(y)| = d(x, y) + d(y, z)$.

Property 3. *The mapping φ has the following properties:*

1) $d(\varphi(x), \varphi(y)) < 2d(x, y)$, for all distinct points $x, y \in X$;

2) if $x, y \in X$ and $d(x, y) = 1$, then $d(\varphi(x), \varphi(y)) \leq d(x, y)$;

3) if $x, y \in \mathbb{N}$ and $x \neq y$, then $d(\varphi(x), \varphi(y)) < d(x, y)$;

4) φ is a continuous mapping.

Proof. Let $x, y \in X$ and $x \neq y$.

If $d(x, y) = 1$, then $d(\varphi(x), \varphi(y)) \leq 1 = d(x, y)$. Assertion 2 is proved.

Assume that $d(x, y) < 1$. We have the following two cases:

Case 1. $x, y \in \mathbb{N}$.

Assume that $x < y$. In this case $d(\varphi(x), \varphi(y)) = \Sigma\{m^{-1} : x + 1 < m \leq y + 1\}$ ¡ $\Sigma\{m^{-1} : x < m \leq y\} \leq d(x, y)$. Moreover, $d(x, y) - d(\varphi(x), \varphi(y)) = |(x + 1)^{-1} - (y + 1)^{-1}|$. Assertion 3 is proved.

Case 2. $x \in B$ *and* $y \in \mathbb{N}$.

Let $x = b_i$, $1 \leq i \leq m$. In this case there exists $n \in \mathbb{N}$ such that $i_{(n-1,m+1)} \leq y \leq i_{(n,m+1)}$ and $d(x, y) = d(b_i, y) = (i_{(n,1)})^{-1} + |f(y) - f(i_{(n,i)})|$.

If $y < i_{(n,i)}$, then $d(\varphi(x), \varphi(y)) = (i_{(n,1)})^{-1} + f(i_{(n,i)}) - f(y+1)$ ¡ $(i_{(n,1)})^{-1} + f(i_{(n,i)}) - f(y)$ $= d(x, y)$.

If $y \geq i_{(n,i)}$, then $d(\varphi(x), \varphi(y)) = i_{(n,1)})^{-1} + f(y + 1) - f(i_{(n,i)} = i_{(n,1)})^{-1} + f(y) - f(i_{(n,i)}) + (y + 1)^{-1} = d(x, y) + (y + 1)^{-1}$. Since $(y + 1)^{-1} < (i_{(n,i)})^{-1} \leq d(x, y)$, we have $d(\varphi(x), \varphi(y)) < 2d(x, y)$. Assertion 1 is proved. Assertion 4 follows from Assertion 1.

Property 4. *If $x \in X$, then $\lim_{n\to\infty} d(\varphi^n(x), \varphi^{n+1}(x)) = 0$.*

Proof. If $x \in B$, then $\varphi(x) = x$ and the assertion is proved. If $x \in \mathbb{N}$, then $d(\varphi^n(x), \varphi^{n+1}(x)) = \Sigma\{z^{-1} : z \leq x+n+1\} - \Sigma\{z^{-1} : z \leq x+n\} = (x+n+1)^{-1}$. Hence $\lim_{n\to\infty} d(\varphi^n(x), \varphi^{n+1}(x)) = \lim_{n\to\infty}(x+n+1)^{-1} = 0$.

Property 5. *The space $(X, \mathcal{T}(d))$ is complete metrizable.*

Proof. If $x \in \mathbb{N}$, then $N_n x = \{x\}$ for each $n \in \mathbb{N}$. If $x = b_i \in B$ and $n \in \mathbb{N}$, then $N_n x = \{x\} \cup \{y \in N : d(x,y) < 2^{-n}\}$. For any $\leq i < j \leq m$ we have $N_1 b_i \cap N_1 b_j = \varnothing$. Then $\mathcal{B} = \{N_n x : x \in X, n \in \mathbb{N}\}$ is a base of open-and-closed subsets of the space $(X, \mathcal{T}(d))$. The proof is complete.

Property 6. *If $x \in \mathbb{N} \subset X$, then $O(x, \varphi) = \{n \in \mathbb{N} : x \leq n\}$. Moreover, if $x, y \in \mathbb{N} \subset X$ and $x < y$, then $O(y, \varphi) \subset O(x, \varphi) \subset O(1, \varphi)$.*

Property 7. *Let $i \leq m$. Then $\lim_{n\to\infty} d(b_i, i_{(n,i)}) = \lim_{n\to\infty}(i_{(n,i)})^{-1} = 0$.*

Property 8. *The space $(X, T(d))$ is not locally compact.*

Proof. Fix $i \leq m$. Assume that U is an open neighbourhood of the point b_i in X. There exists $k \in \mathbb{N}$ such that $\{x \in X : d(b_i, x) < 2k^{-1}\} \subset U$. For each $n \geq k$, fix $x_n \in I_{(n,i)}$ such that $k^{-1} \leq d(i_{(n,i)}, x_n) < 2k^{-1}$. Then $\{x_n \in \mathbb{N} : n \in \mathbb{N}, n \geq k\}$ is a closed discrete sequence of the space $(X, \mathcal{T}(d))$ such that $x_n \in U$ and $x_n < x_{n+1}$ for each $n \in \mathbb{N}$, $n \geq k$.

Property 9. *All points $x \in B$ are points of accumulation of the Picard orbit $O(n, \varphi)$, $n \in \mathbb{N}$.*

Property 10. *The Picard orbit $O(x, \varphi)$ is not convergent in (X, d), for any $x \in \mathbb{N}$.*

Bibliography

1. Berinde V. and Choban M. Remarks on some completeness conditions involved in several common fixed point theorems. Creat. Math. Inform. 19, no. 1, 2010. p. 1–10.
2. Berinde V. and Choban M. Generalized distances and their associate metrics. Impact on fixed point theory. Creat. Math. Inform. 22, no. 1, 2013. p. 23–32.
3. Chittenden E. W. On the equivalence of écart and voisinage. Trans. Amer. Math. Soc. 18, 1917. p. 161–166.
4. Choban M. Fixed points for mappings defined on pseudometric spaces. Creat. Math. Inform. 22, no. 2, 2013. p. 173–184.
5. Choban M. Selections and fixed points theorems for mapping defined on convex spaces. ROMAI J. 10, no. 2, 2014. p. 11–44.
6. Choban M. Fixed points for mappings defined on generalized gauge spaces. Carpathian J. Math. 31, no. 3, 2015. p. 313–324.
7. Choban M. Fixed points of mappings defined on spaces with distance. Carpathian J. Math. 32, no. 2, 2016. p. 173–188.
8. Choban M. and Berinde V. Two open problems in the fixed point theory of contractive type mappings on quasimetric spaces. Carpathian J. Math. 33, No. 2, 2017. p. 169–180.
9. Engelking R. General Topology. PWN, Warszawa, 1977.
10. Granas A. and Dugundji J. Fixed Point Theory. Berlin: Springer, 2003.

11. Nedev S. Y. O-metrizable spaces. (Russian) Trudy Moskov. Mat. Ob-va 24, 1971. p. 201–236. (English translation: Trans. Moscow Math. Soc. 24, 1974. p. 213–247).
12. Niemytzki V. On the third axiom of metric spaces. Trans Amer. Math. Soc. 29, 1927. p. 507–513.
13. Niemytzki V. Über die Axiome des metrischen Raumes. Math. Ann. 104, 1931. p. 666–671.
14. Rus I. A., Petruşel A. and Petruşel G. Fixed Point Theory. Cluj-Napoca: Cluj University Press, 2008.
15. Shrivastava R., Ansari Z. K. and Sharma M. Some results on fixed points in dislocated quasi-metric spaces. J. Adv. Stud. Topol. 3, no. 1, 2012. p. 25–31.
16. Wilson W. A. On semi-metric spaces. Amer. J. Math. 53, no. 2, 1931. p. 361–373.
17. Wilson W.A. On quasi-metric spaces. Amer J. Math. 53, no. 6, 1931. p. 675–684.