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THE PROBLEM OF THE NUMBER π AND ANOTHER CONSTRUCTION OF TRIGONOMETRY

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Abstract. In this paper, the solution of the problem of the number π has been described. A definition of this number was formulated according to the model of the definition of the number e, mathematically well understood. Then this number was based on the definition of the length of the circle and of the arcs of the circle, and as well as on the definition of trigonometric functions of real variable.

Keywords: number π, circle length, absolute trigonometry.

PROBLEMA NUMĂRULUI π ŞI ALTĂ CONSTRUCŢIE A TRIGONOMETRIEI

Rezumat. În această lucrare se descrie rezolvarea problemei numarului π. S-a formulat definiţia acestui număr după modelul definiţiei numarului e, bine inchegată din punct de vedere matematic. Apoi acest numar a fost pus la baza definiţiei lungimii cercului şi arcelor de cerc, precum şi la baza definiţiei funcţiilor trigonometrice de variabilă reală.

Cuvinte-cheie: numărul π, lungimea cercului, trigonometria absolută.

1. Introduction

In this paper the oldest and most controversial mathematical problem in the history of mankind is studied. Thousands of years ago the barrels builders observed that between the length of the circle and its diameter there is a ratio that does not depend on the length of the diameter. This ratio today is known as the "number π". Famous stories about the problem of the number π are contained in the books [1, 2, 3, 4, 6, 9]. Problem of the ratio π it has become an enigma for mathematicians of all times. Over the years for this ratio chronologically were proposed different numbers that approximate it. The approximation of π is found in the Old Testament as $\pi = 3$, and in the Ancient Chinese used the inaccurate value $\pi = 3$. The Babylonians used the estimation 25/8. In the Egyptian Rhind Papyrus (around 1650 BC, copied by the scribe Ahmes from a document dated to 1850 BC), in the formula for the area of a circle, treats π as $(\frac{16}{9})^2 \approx 3.16$.

Antiphon of Rhamnus (480 - 411 BC) and Bryson of Heraclea (5th century BC) used the inscribed and circumscribed polygons and found an approximate method of solving the problem of squaring the circle. Squaring the circle is a famous problem proposed by ancient geometries whose solution depends on the nature of the number π. The Ancient mathematician Archimedes of Syracuse (287 - 212 BC) is considered to be the first who proposed an effective method, the polygonal approach, to estimate the value of π. Archimedes proved that $223/71 < \pi < 22/7$. Mathematicians, using polygonal algorithms, obtained distinct approximations of π. The Chinese mathematician Liu Hui iu Hui (3rd century CE) obtained a value of π of 3.1416, the Chinese mathematician Zu Chongzhi (around 480 AD) determined that $3.1415926 < \pi < 3.1415927$, the Italian mathematician Fibonacci of Pisa (c. 1170 - c.

1240-50) employed the value $\pi \approx 3 + \frac{\sqrt{2}}{10} \simeq 3.14142$, in 1593 the French mathematician Francois Viete (1540 - 1603) discovered the first infinite product in the history of mathematics by giving an expression of π, named Viete's formula:

$$\pi = 2 \times \frac{2}{\sqrt{2}} \times \frac{2}{\sqrt{2+\sqrt{2}}} \times \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \times \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} \times \cdots.$$

In 1655 the English mathematician John Wallis (1616 - 1703) discovered the second infinite product:

$$\frac{\pi}{2} = \left(\frac{2}{1}\cdot\frac{2}{3}\right)\cdot\left(\frac{4}{3}\cdot\frac{4}{5}\right)\cdot\left(\frac{6}{5}\cdot\frac{6}{7}\right)\cdot\left(\frac{8}{7}\cdot\frac{8}{9}\right)\cdots.$$

The Welsh mathematician William Jones (1675 - 1749) in 1706 used the symbol π to represent the ratio of the circumference of a circle to its diameter. Leonhard Euler (1707 - 1783) started using π form beginning with his "Essay Explaining the Properties of Air" (1727). The Euler's Identity $e^{i\pi} + 1 = 0$ is one of the simplest and most elegant equations. Euler established that

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

Distinct infinite representation of the number π were proposed by many mathematicians and, in particular, by English scientist Isaac Newton (1642 - 1726), German mathematician Gottfried Wilhelm Leibniz (1646 - 1716), Scottish mathematician James Gregory (1638 - 1675). In 1706 professor of astronomy John Machin (1686 - 1751) at Gresham College, London, in 1706 using the formula $\frac{\pi}{4} = arctg\frac{1}{5}$ - $arctg\frac{1}{239}$ computed π to 100 decimal places. Other mathematicians created variants, now known as Machin-like formulae, for calculating digits of π.

Swiss mathematician Johann Heinrich Lambert (1728 -1777) in 1761 proved that the number π is irrational, in 1794 French mathematician Adrien-Marie Legendre (1752 - 1833) proved that π^2 is also irrational. Finally, in 1882 German mathematician Ferdinand von Lindemann (1852 - 1939) proved that π is transcendental. Hence it is impossible to square the circle by compass and straightedge. Moreover, the number π is not the root of some polynomial with rational coefficients and, in particular, cannot be obtained from the rational numbers with the help of operations $+, -, \times, :$ and the extraction of the root of any order.

The development of computers in the mid-20th century revolutionized the hunt for digits of π. There are distinct motives for computing the number π: cosmological calculations for the circumference of the observable universe; the number π it is found in many formulae from the fields of geometry, trigonometry, statistics, physics, Fourier analysis, and so.

This article includes introduction, conclusions and four sections. In the Section 2 we present a new definition of the number π. In Section 3 we calculate the length of the circle and the circle arcs. In Section 4 we presented the definition of trigonometric functions of real variables. In Section 5 we deduce the formulas of the algebraic sum of two real numbers for the trigonometric functions *cosine* and *sine*. Sections 3 - 5 are aspect methodological too.

2. A new definition of the number π

Let $\mathbb{R}$ be the field of reals, $\mathbb{Q}$ be the subfield of rationals and $\mathbb{I} = [0, 2^{-1}]$. Consider the function $f : I \longrightarrow \mathbb{R}$, where $f(r) = sin(r180^0)$ for any $r \in \mathbb{I}$.

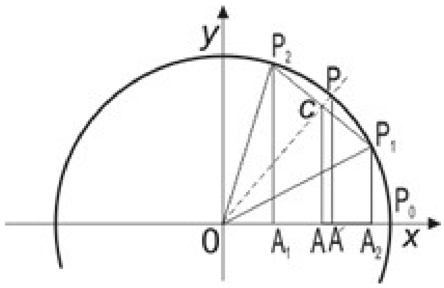

Figure 1.

We then consider a trigonometric circle, fig.1, and note with P_0 the point with the coordinates $(1; 0)$, the intersection point of the circle with the positive semi-axis Ox. For the rational numbers r_1 and r_2 from $\mathbb{I}$, $0 \leq r_1 < r_2 \leq \frac{1}{2}$, we associate the points P_1 and P_2 on the trigonometric circle, so that the arcs $P_0P_1 = r_1 180^0$ and $P_0P_2 = r_2 180^0$ are oriented counterclockwise. We carry the chord P_1P_2 and the segments $A_2P_1 = sin(r_1 180^0)$, and $A_1P_2 = sin(r_2 180^0)$. As a result we obtain the trapezoid $A_1A_2P_1P_2$, where AC is the mid line. The semi-right OC passes through the middle of the chord P_1P_2 and so is the bisector of the angle P_1OP_2 and passes through the middle P of the arc P_1P_2. It is easy to believe that the arc P_0P contains $\frac{r_1+r_2}{2}180^0$. Indeed, from fig.1 we deduce the following: $\angle P_0OP = \angle P_0OP_1 + \angle P_1OP$ and $\angle P_0OP = \angle P_0OP_2$ - $\angle POP_2$. We gather together these equality and taking into account that $\angle P_1OP = \angle POP_2$, we obtain $2\angle P_0OP = \angle P_0OP_1 + \angle P_0OP_2 = (r_1 + r_2)180^0$ and so $\angle P_0OP = \frac{r_1+r_2}{2}180^0$. From $OC < OP$ it follows that the segment $AC < A'P = sin(\frac{r_1+r_2}{2}180^0)$ and therefore $\frac{sin(r_1 180^0)+sin(r_2 180^0)}{2} = \frac{A_2P_1+A_1P_2}{2} = AC < A'P = sin(\frac{(r_1+r_2)}{2} \cdot 180^0)$. Thus, for $0 \leq r_1 < r_2 \leq \frac{1}{2}$ we obtain

$$sin(r_1 180^0) + sin(r_2 180^0) < 2sin(\frac{(r_1+r_2)}{2} \cdot 180^0). \quad (1)$$

From inequality (1) we deduce that the function $f(r) = sin(r180^0)$ is concave. According to the concavity criterion, the function $f_1(r) = \frac{sin(r180^0)-sin0}{r-0} = \frac{1}{r} \cdot sin(r180^0)$ is decreasing on the set $(0, 2^{-1}]$. For $r = p^{-1}$, $p \in \mathbb{R}$, $p \geq 2$, we obtain the function $f_2(p) = psin(\frac{180^0}{p})$ which is increasing on the sets $\mathbb{Q}_2 = \{p \in \mathbb{Q} : p \geq 2\}$ and $\mathbb{R}_2 = \{p \in \mathbb{R} : p \geq 2\}$.

We consider $p = n \geq 2$, $n \in \mathbb{N} = \{1, 2, ...\}$, and obtain the strictly increasing string $\{n \cdot sin(\frac{180^0}{n}) : n \in \mathbb{N}\}$. The perimeter of a regular polygon with n sides inscribed in a circle is $2Rn \cdot sin(\frac{180^0}{n})$, and the perimeter of a square circumscribed by the circle is $8R$. Thus we have inequality $2Rn \cdot sin(\frac{180^0}{n}) < 8R$, from which it follows that $n \cdot sin(\frac{180^0}{n}) < 4$. So the string $\{n \cdot sin(\frac{180^0}{n}) : n \in \mathbb{N}\}$ is strictly increasing and marginally superior and therefore convergent. We note the limit of this string with the letter $\pi = lim_{n\to\infty} n \cdot sin(\frac{180^0}{n})$.

Since the perimeter of a regular polygon with n sides inscribed in a circle is $C_n = 2Rn \cdot sin(\frac{180^0}{n})$ and $lim_{n\to\infty} C_n$ is th circumference C of the circle, we obtain that $C = lim_{n\to\infty} 2Rn \cdot sin(\frac{180^0}{n}) = 2R\pi$. By the definition, the number $C = 2R\pi$ represents the length of the circle.

With the formulated definition, the puzzle generated by the problem of the number π was put to an end, and we obtained a new solution of one of the oldest and most controversial mathematical problems in the history.

Remark. The mechanism elaborated in this paper for the definition of the number π can also be applied to the definition of the number e.

Indeed, the logarithmic function in base 10 is concave and increasing throughout its definition field. We consider the arbitrary values $x_1 > -1$ and $x_2 > -1$ and obtain

$lg(1 + x_1) + lg(1 + x_2) = 2\ lg\sqrt{(1 + x_1)(1 + x_2)} < 2lg\frac{(1+x_1)+(1+x_2)}{2}$.

Thus, the function $lg(1 + x)$ given on the interval $(-1, \infty)$ is concave. According to the concavity criterion the function $g(x) = \frac{lg(1+x)-lg1}{(1+x)-1} = lg(1 + x)^{(1/x)}$ is decreasing on the perforated interval $(-1, \infty)$, $x \neq 0$. It follows that the function $g_1(x) = (1 + x)^{1/x}$ is also decreasing on this perforated interval, and on the interval $(0, +\infty)$ is bounded higher by the value $g_1(-1/2) = 4$. Therefore at the point $x = 0$ there is the right lateral limit of the function $g_1(x)$ which is denoted by the letter e.

Thus by definition $e = lim_{x\to 0^+}(1+x)^{(1/x)}$ or for $x = \frac{1}{n}$, $n \in \mathbb{N}$, we have $e = lim_{n\to\infty}(1+\frac{1}{n})^n$. Thus these two real numbers π and e are discovered in different historical millennia, have entered the set of real numbers by the same mechanism, and both have been placed - one at the basis of trigonometric functions, and another at the basis of hyperbolic functions.

3. The length of the circle and the circle arcs

By the definition, the number $C = 2R\pi$ represents the length of the circle of radius R. Next we consider a circle arc of radius R and size α^0, fig.2.

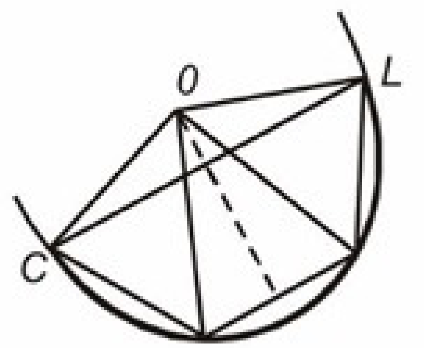

Figure 2.

Then we write in this circle arc a regular polygonal line with n sides. The perimeter l_n of this polygonal line is $l_n = = 2Rn \cdot sin(\frac{\alpha^0}{2n})$. In the obtained formula we substitute: $\frac{\alpha^0}{2n} = \frac{180^0}{p}$. Hence $2n = \frac{\alpha^0}{180^0} \cdot p$ and the perimeter l_n takes the form: $l_n = l_p = \frac{\alpha^0}{180^0} Rpsin(\frac{!80^0}{p}) = \frac{\alpha^0 R}{180^0}(psin(\frac{!80^0}{p}))$, where the function $psin(\frac{180^0}{p})$ is ascending and contains the convergent string to the number π. So there is the finite limit l of the perimeter l_n of the polygonal line and $l = lim_{n\to\infty} l_n = lim_{p\to\infty} l_p = \frac{\alpha R}{180} \cdot lim_{p\to\infty}(psin(\frac{180^0}{p})) = \frac{\alpha R}{180} \cdot \pi = \pi R \frac{\alpha}{180}$.

Therefore, by definition, the length l of a circle arc of radius R and size α^0 is calculated from the formula

$$l = \pi R \frac{\alpha}{180}. \quad (2)$$

From (2) it follows that $\alpha^0 = \frac{l}{\pi R} \cdot 180^0$ and $\frac{l}{2\pi R} = \frac{\alpha^0}{360^0}$.

From the last equality, adopted by consensus in the school textbooks, results the formula (2).

4. Definition of trigonometric functions of real variables

In our definition of the number π we used the function sin. Next we will present the definition of the functions cos and sin using the trigonometric circle. We will first express any real number x by the number π. We consider the number x and note by $n(x)$, $n(x) \in \mathbb{Z}$ $= \{0, \pm1, \pm2, \cdots\}$, the whole part of the fraction $\frac{x}{2\pi}$, and the fractional part of the number $\frac{x}{2\pi}$ is denoted by $\theta(x)$, $0 \leq \theta(x) < 1$. Thus, from $\frac{x}{2\pi}$ - $n(x) = \theta(x)$ we obtain the equality $x = 2n(x)\pi + 2\theta(x)\pi$. After the representation of the number x by the numbers $n(x)$ and $\theta(x)$, we consider a trigonometric circle and note with $P_0(1;0)$ the intersection point of the circle with the positive semi-axis Ox, fig.3.

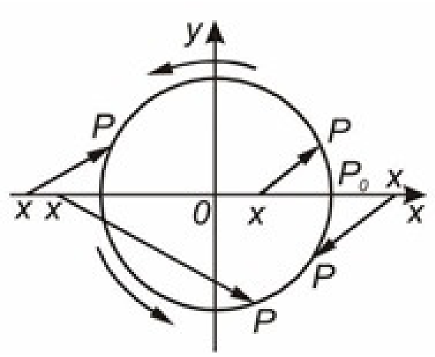

Figure 3.

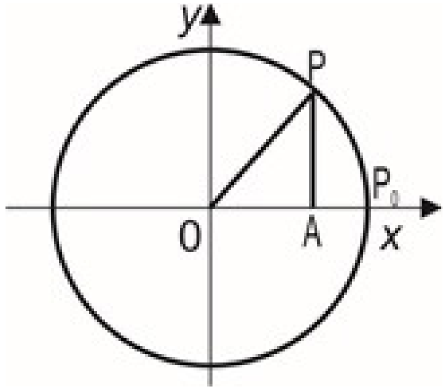

Figure 4.

We then define an application (a mapping) of the set of real numbers in the set of points of the trigonometric circle according to the following rule: the arbitrary number $x \in \mathbb{R}$ we correspond to the point $P(u;v)$ on the trigonometric circle so that the arc P_0P is oriented counterclockwisein and the length of this arc is equal to $2\theta(x)\pi$, i.e. $\smile (P_0P) = 2\theta(x)\pi$. According to this rule each real number x corresponds to a well-determined value of the abscissa u and a well-determined value of the ordinate v of the point $P(u;v)$. Therefore both the abscissa u and the ordinate v of the point $P(u;v)$ are functions of real variable x. The function $u : \mathbb{R} \to \mathbb{R}$, where $u = cosx$, is called the trigonometric cosine. The function $v : \mathbb{R} \to \mathbb{R}$, where $v = sinx$, is called trigonometric sine, $x \in \mathbb{R}$. From the definition of the functions sinus and cosine, we have $|sinx| \leq 1$, $|cosx| \leq 1$ and identity $sin^2x + cos^2x = 1$. The periodicity property also results - the $sinx$ and $cosx$ functions are periodic and their general period is $2n\pi$, $n \in \mathbb{Z}$, and the main period is 2π. From the formula $\alpha^0 = \frac{l}{\pi R} \cdot 180^0$, for $R = 1$ and $l = 2\theta\pi$ it turns out that $\alpha^0 = 2\theta 180^0$. Thus, the arc of length $\smile (P_0P) = 2\theta\pi$ contains $2\theta 180^0$. In this case from Fig. 4 it is easy to calculate the abscissa and the ordinate of the point $P(u;v)$. Indeed $u = OA = cos(2\theta\pi) = cos(2\theta 180^0)$ and $v = AP = sin(2\theta\pi) = sin(2\theta 180^0)$.

Example. $sin\frac{5}{6}\pi = sin(2\frac{5}{12}180^0) = sin150^0 = \frac{1}{2}$.

It is easy to prove that the function $cosx$ is even, and the function $sinx$ is odd. For this we consider a trigonometric circle, fig.5. The intersection point of the circle with the positive semi-axis Ox is also denoted by $P_0(1;0)$. The number $x = 2n\pi + 2\theta\pi$ corresponds the point $P_1(cosx;sinx)$ to the trigonometric circle, and the number $-x = -2n\pi$ - $2\theta\pi$ = $-2(n+1)\pi + 2(1-\theta)\pi$ corresponds the point $P_2(cos(-x);sin(-x))$ to the circle. Since

$\smile (P_0P_1) = 2\theta\pi$ and $\smile (P_0P_2) = 2(1-\theta)\pi = 2\pi - 2\theta\pi$, it turns out that $\smile (P_0P_1) + \smile (P_0P_2) = 2\pi$ and $\smile (P_2P_0) + \smile (P_0P_2) = 2\pi$. Therefore $\smile (P_0P_1) = \smile (P_2P_0)$ and so the points P_1 and P_2 are symmetrical with respect to the axis Ox, where it follows that $cos(-x) = cosx$, and $sin(-x) = -sinx$.

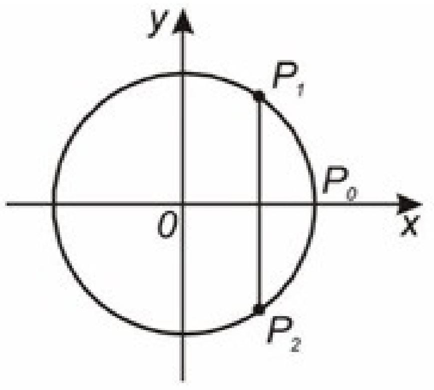

Figure 5.

5. The cosine and sine of the algebraic sum of two real numbers

The most rational way to establish the relations between the trigonometric functions sinus and cosine begins with the cosine formula of the sum of two real numbers.

We consider a trigonometric circle, fig. 6, where $P_0(1,0)$ is the intersection point of the circle with the positive semi-axis Ox.

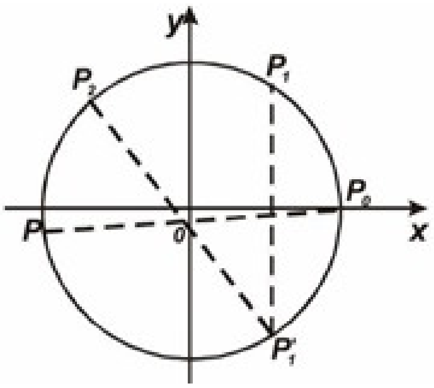

Figure 6.

Now we consider the arbitrary real numbers x, t and their sum $x+t$. We represent these numbers by the number π as follows: $x = 2n\pi + 2\theta_1\pi$, $t = 2k\pi + 2\theta_2\pi$ and $x+t = 2m\pi + 2(\theta_1+\theta_2)\pi$, where $0 \leq \theta_1 < 1$, $0 \leq \theta_2 < 1$ and $m, n, k \in \mathbb{Z}$. According to the definition of the functions sinus and cosine the real numbers x and t are associated with the points $P_1(cosx, sinx) = P_1(cos2\theta_1\pi, sin2\theta_1\pi)$ and $P_2(cost, sint) = P_2(cos2\theta_2\pi, sin2\theta_2\pi)$ of the trigonometric circle respectively, and the sum $x+t$ corresponds to the point $P(cos(x+t), sin(x+t)) = P(cos2(\theta_1+\theta_2)\pi, sin2(\theta_1+\theta_2)\pi)$ such that the arcs P_0P_1, P_0P_2 and P_0P are oriented counterclockwise and the lengths of the arcs P_0P_1 and P_0P_2 are as follows: $\smile P_0P_1 = 2\theta_1\pi$ and $\smile P_0P_2 = 2\theta_2\pi$.

If $0 < \theta_1 + \theta_2 < 1$, which occurs at least in cases when P_1 and P_2 belong to the semicircle located in quadrants 1 and 2, then the length of the arc P_0P is represented by the formula:

$$\smile (P_0P) = 2(\theta_1 + \theta_2)\pi = 2\theta_1\pi + 2\theta_2\pi = \smile (P_0P_1) + \smile (P_0P_2).$$

If, however, $1 \leq \theta_1 + \theta_2 < 2$, which occurs at least in cases where the points P_1 and P_2 belong to the semicircle located in quadrants 3 and 4, then $\smile (P_0P) = 2(\theta_1 + \theta_2)\pi = 2(\theta_1 + \theta_2 - 1)\pi + 2\pi$, where $0 < \theta_1 + \theta_2 - 1 < 1$. In this case the arc P_0P covers the circle, but so is represented the sum $\smile (P_0P_1) + \smile (P_0P_2) = 2\theta_1\pi + 2\theta_2\pi = 2(\theta_1 + \theta_2 - 1)\pi + 2\pi$. So in this case we have $\smile (P_0P) = \smile (P_0P_1) + \smile (P_0P_2)$ too.

Now we consider on the unit circle the point $P'_1(cosx, -sinx)$, symmetrical with the point $P_1(cosx, sinx)$ with respect to the axis Ox and so $\smile (P'_1P_0) = \smile (P_0P_1)$. Thus

$$\smile (P_0P) = \smile (P_0P_1) + \smile (P_0P_2) = \smile (P'_1P_0) + \smile (P_0P_2) = \smile (P'_1P_2).$$

From the equality of the lengths of the arcs P_0P and P'_1P_2 results the equality of the respective chords. So

$$P_0P = \sqrt{(cos(x+t)-1)^2 + sin^2(x+t)} = \sqrt{(cosx - cost)^2 + (sinx + sint)^2} = P'_1P_2.$$

From this equality we obtain: $cos(x + t) = cosxcost - sinxsint$.

We substitute t with -t and obtain: $cos(x - t) = cosxcost + sinxsint$.

In order to deduce the formula of the *sine* of the sum, we will use the coordinates of the intersection point of the circle trigonometric with positive semi-axis Oy. These are $u = cos\frac{\pi}{2} = 0$ and $v = sin\frac{\pi}{2} = 1$.

Also, to deduce some relations between the functions *sinus* and *cosine*, we will use the coordinates of the intersection point of the circle trigonometric with negative semi-axis Ox. These are $u = cos\pi = -1$ and $v = sin\pi = 0$.

Therefore $cos(\frac{\pi}{2} - x) = cos\frac{\pi}{2}cosx + sin\frac{\pi}{2}sinx = sinx$. In this formula we substitute x with $\frac{\pi}{2} - x$ and we obtain $sin(\frac{\pi}{2} - x) = cos(\frac{\pi}{2} - (\frac{\pi}{2} - x)) = cosx$.

We now deduce the formula of the sum of the sine: $sin(x + t) = cos(\frac{\pi}{2} - (x + t)) = cos((\frac{\pi}{2} - x) - t) = cos(\frac{\pi}{2} - x)cost + sin(\frac{\pi}{2} - x)sint = sinxcost + cosxsint$. Thus $sin(x+t) = sinxcost + cosxsint$. We substitute t with $-t$ and obtain: $sin(x - t) = sinxcost - cosxsint$.

From the established formulas, we deduce the other relations between *sine* and *cosine*. For example, we have $sin(\pi + t) = sin\pi cost + cos\pi sint = -sint$.

In his work [8] the author sets out a solution to this problem, but too voluminous.

6. Conclusions

In the exposed work number π was defined as the limit of the well determined string, using the circumference of a circle and the function *sine* for the small angles. Then, this number was the basis for defining the trigonometric functions of a real variable. We also mention the fact that the mechanism elaborated in this work opens the same path that enters the set of real numbers, both number π and number e, although these two numbers have appeared in history, in different millennia and in different circumstances.

Some of the theses of this work have been presented by the author at the international conference CAIM-18 in September 2018 [7].

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