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A note on some open problems in topological algebra

A. V. Arhangel'skii

Abstract. One of the old open problem of C_p-theory, where I is the unit segment is the following question: Are the spaces $C_p(I)$ and $C_p(I^2)$ homeomorphic? In the present article distinct open problems of the topological algebra are examined

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Privitor la unele probleme nerezolvate ale algebrei topologice

Rezumat. Una dintre vechile probleme deschise ale teoriei C_p, unde I este segmentul unitate, este urmatoarea întrebare: Sunt oare homeomorfe spaţiile $C_p(I)$ şi $C_p(I^2)$? În prezentul articol sunt examinate probleme deschise distincte ale algebrei topologice

Cuvinte cheie: grup topologic, spaţiu omogen, secvenţă convergentă, cvasiretracţie.

Objects of topological algebra, defined as a certain combination of algebraic and topological structures, often give rise to original and unusual questions. A special additional topological property of many topological spaces of this kind is homogeneity. A topological space X is called *homogeneous* if, for any x, y in X there exists a homeomorphism f of X onto itself such that $f(x) = y$ and $f(X) = X$. Clearly, all topological groups, in particular, all linear topological spaces are homogeneous. This simple fact provides us with a natural way to construct many homogeneous compact spaces, since there are many compact topological groups. Some of them are non-metrizable. This occurs precisely when a compact topological group is not sequential - that is, when its topology cannot be described in terms of convergent sequences. In this connection, it is especially interesting that every infinite compact topological group has many non-trivial convergent sequences. But the following question, posed by Walter Rudin [7, 8] more than 60 years ago, is still open:

Problem 0.1. *(W. Rudin) Is it true that every infinite homogeneous compact Hausdorrf space contains a non-trivial convergent sequence?*

Many compact topological groups contain, in fact, dense sequential subgroups. In this connection I have formulated, about forty years ago, the next question, which seems to be still not answered (see [1]):

Problem 0.2. *Is it true that every infinite compact topological group contains a dense sequential subspace?*

However, the next statement holds (see [5]):

Theorem 0.3. *Under CH, every homogeneous sequential compact Hausdorff space is first countable, and hence, its cardinality does not exceed* 2^{ω}.

In this connection, the following questions arise:

Problem 0.4. *Is it true in ZFC that every homogeneous sequential compact Hausdorff space is first countable?*

I also want to mention another open question [1]:

Problem 0.5. *Suppose that X is a paracompact p-space. Then is its free topological group $F(X)$ (or the Abelian version of it) paracompact?*

It had been shown in [1] that if X is metrizable, then the answer to the last question is "yes".

Here are some open problems for linear topological spaces over the field R of real numbers. First of all, locally convex infinite dimensional linear topological spaces of this kind should be considered. We have them in mind below. We use 0 to denote also the zero vector of such spaces.

Suppose that L is a linear topological space. Put $L_0 = L \setminus \{0\}$. Suppose also that bL is a Hausdorff compactification of L, and that Y is the remainder $bL \setminus L$.

For each $x \in L_0$ and each $n \in \omega$, put $B_{x,n} = \overline{\{\alpha x : \alpha \in R, \alpha > n\}}$, $B_x = \cap\{B_{x,n} : n \in \omega\}$, and $Y_x = B_x \cap Y$.

A mapping f of the subspace L_0 of L into Y will be called a *quasiretraction* of L_0 into Y, if f is continuous and $f(x) \in Y_x$, for each $x \in L_0$.

In this general setting, it is natural to ask the next basic question:

Problem 0.6. *For which linear topological spaces L there exists a Hausdorff compactification bL of L such that there exists a quasiretraction of L_0 into Y?*

Any space L satisfying the condition in the last problem can be called a *rain space*. In the next question we use the notation described in the preceding question and before it.

Problem 0.7. *Suppose that $L = R^\tau$, where τ is an infinite cardinal, and let Y be the Stone-Čech remainder $\beta L \setminus L$ of L. Then is it true that there exists a quasiretraction of L_0 into Y?*

The next double question is closely related to the preceding problem, but is formulated in much simpler terms.

Problem 0.8. *We again use the notation described in the preceding two questions. Suppose that $L = R^\tau$, where τ is an infinite cardinal, and that bL is the Stone-Čech compactification βL of L. Then is it true that there exists a continuous mapping of the space $L_0 = L \setminus \{0\}$ onto some dense subspace of the space $\beta L \setminus L$? Is it true that there exists a continuous mapping of the space L onto some dense subspace of the space $\beta L \setminus L$?*

The last open problem in this short list is more than 30 years old. See [2] for one of the early appearances of it in print and for some related questions and references. For a Tychonoff space X, $C_p(X)$ denotes the space of continuous real-valued functions on X endowed with the topology of pointwise convergence (see [2]).

Problem 0.9. *Let I be the closed unit interval, I^2 be the square of I, and K be the Cantor set, all taken with the usual topologies. Then we have the following three simply formulated questions:*

a): Are the spaces $C_p(I)$ and $C_p(I^2)$ homeomorphic?

b): Are the spaces $C_p(I)$ and $C_p(K)$ homeomorphic?

c): Are the spaces $C_p(K)$ and $C_p(I^2)$ homeomorphic?

Note that the the answers to the last three questions are in the negative for linear homeomorphisms. This follows from a deep theorem of V.G. Pestov, see [6].

For more problems and references on topological problems in topological algebra, see [3], [4], and [5].

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(Alexander V. Arhangel'skii) MPGU, Moscow, Russian Federation

E-mail address: `arhangel.alex@gmail.com`